振動解析の基礎

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序

東日本大地震が発生した。４年以内に震度７クラスの地震が都内で起こる確率が７０％になると東京大学地震研究所の平田直教授から発表された。東日本大地震以降M５クラスの地震発生頻度が東日本大地震発生前に比べて、高くなった（気象庁 http://www.seisvol.kishou.go.jp/eq/2011_03_11_tohoku/aftershock/）。

斯かる状況下、振動制御装置、免震装置などの研究も盛んに行われているようである。斯かる発明を担当する弁理士を多いであろう。当該発明を処理するに当たって、振動解析の基礎知識が必要となる。

そこで、振動制御に必要な振動解析のために必要な基本的な知識を記載した。行列の偏微分方程式を見ただけでアレルギーを起こす人もいるかもしれないが、要は固有値問題である。

仕事をする上で参考になれば幸甚である。

1. Single Degree Of Freedom System

Consider Single Degree Of Freedom System ("S.D.O.F.") as shown in Fig. 1.

(1) Equation Of Motion

$f(t) = P \cdot Re(e^{i\omega t})$ f(t) acting to subject is

(1)

Due to D'Alembert's principle, equation of motion is

$$m\ddot{x} + c\dot{x} + kx = P \cdot Re(e^{i\omega t})$$

$$\ddot{x} + 2\beta\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m} \cdot Re(e^{i\omega t})$$

ω_0 : natural circular frequency

β : damping constant

$$\dot{x} = \frac{\partial x}{\partial t}, \quad \ddot{x} = \frac{\partial^2 x}{\partial t^2}$$

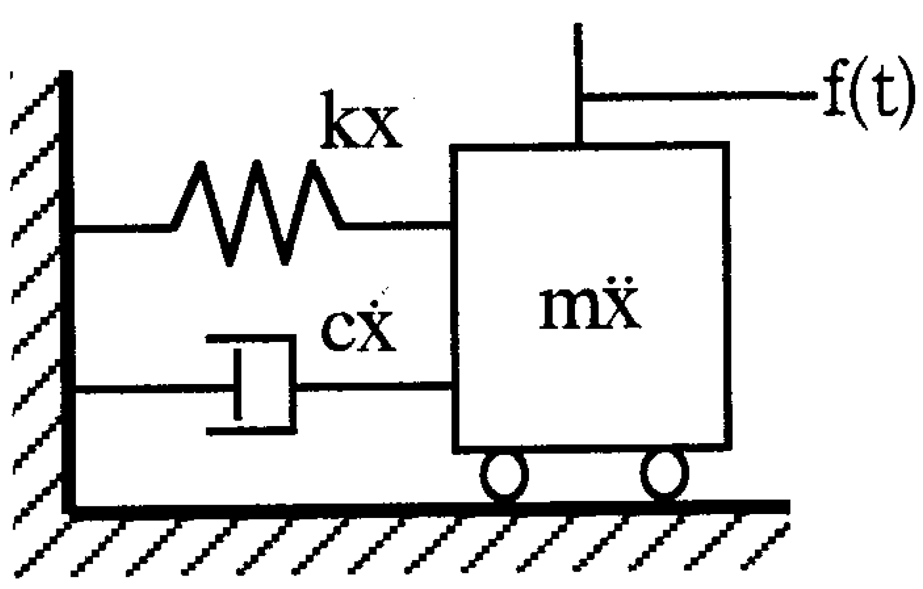


Fig. 1

(2) Response

General solution x

$$x = x_h + x_p$$

x_h : homogenious solution, x_p : particular solution

$$x_h = e^{-\beta\omega_0 t}(a_1 \cos\omega_d t + a_2 \sin\omega_d t), \qquad \omega_d = \sqrt{1-\beta^2}\,\omega_0 \quad \text{in case of } \beta \le 1.$$

Assume x_p is

$$x_p(t) = Ae^{i\omega t}$$

substituting x_p into equa. (1)

$$A = \frac{1}{m(\omega_0^2 - \omega^2 + 2i\beta\omega_0\omega)} \equiv H(i\omega)$$

$H(i\omega)$: complex frequency response function

$$A = |H(i\omega)| e^{i\varphi(\omega)}$$

where $\varphi(\omega) = -\mathrm{Arg}H(i\omega)$, which is phase shift.

$$x_p(t) = |H(i\omega)| e^{i(\omega t - \varphi(\omega))} = \frac{1}{m\sqrt{\left\{(\omega_0^2 - \omega^2)^2 + (2i\beta\omega_0\omega)^2\right\}}} e^{i(\omega t - \varphi(\omega))}$$

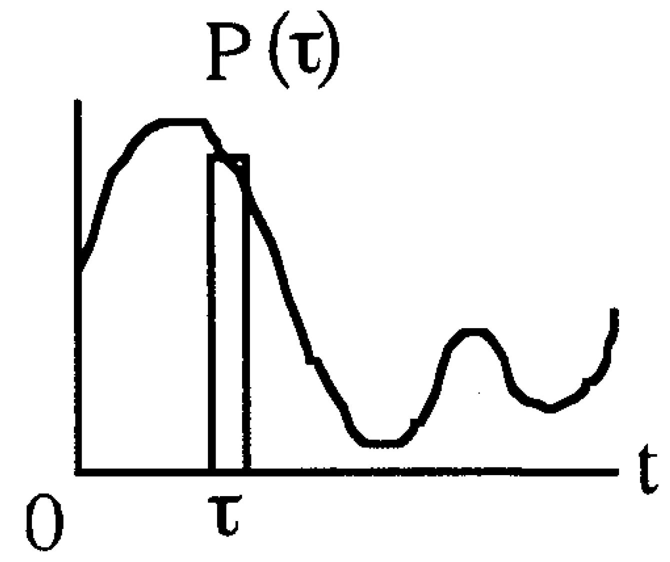
$$\therefore \; x = e^{-\beta\omega_0 t}(a_1 \cos\omega_d t + a_2 \sin\omega_d t) + \frac{1}{m\sqrt{\left\{(\omega_0^2 - \omega^2)^2 + (2i\beta\omega_0\omega)^2\right\}}} e^{i(\omega t - \varphi(\omega))}$$

Response To Transient Force

Above external force is periodic force. In case of transient force, x is indicated as follows, which is called to "unit response function".

$$x = \frac{1}{m\omega_d} e^{-\beta\omega_0 t} \sin\omega_d t$$

$$x_p = \frac{1}{m\omega_d} \int_0^t P(\tau) e^{-\beta\omega_0 (t-\tau)} \sin\omega_d (t-\tau) d\tau$$



$$\therefore \; x = e^{-\beta\omega_0 t}(a_1 \cos\omega_d t + a_2 \sin\omega_d t) + \frac{1}{m\omega_d} \int_0^t P(\tau) e^{-\beta\omega_0 (t-\tau)} \sin\omega_d (t-\tau) d\tau$$

2. Two Degrees Of Freedom System

Next, I will describe how to derive vibration equation of Two Degrees Of Freedom System (hereinafter called "T.D.O.F.") due to Lagrangean Equation.

(1) Lagrangean Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} + \frac{\partial F}{\partial \dot{x}_i} - Q_i = 0 \qquad i = 1,2,3,4,5\cdots,\cdots$$

$i = 1,2$ when two-degree-of-freedom system

$L = T - V$

T : kintic energy, V : potencial energy, F : dissipation, Q : genealized energy,

here $u = u(\eta,t) = \psi(\eta) \cdot x(t)$

(2) Rayleigh-Ritz approach to tower-equipment system

I will consider a model as shown in Fig. 2. It is assumed that the tower has mainly bending, and the observation deck (η=L) has mainly rigid-body translation. The equipment at η=L is modelled single-degree of freedom oscillator with parameters m, c, k.

Assuming that the vibration shape of the tower is

$$\psi(\eta) = \frac{1}{11}\left\{20\left(\frac{\eta}{L}\right)^2 - 10\left(\frac{\eta}{L}\right)^3 + \left(\frac{\eta}{L}\right)^5\right\}$$

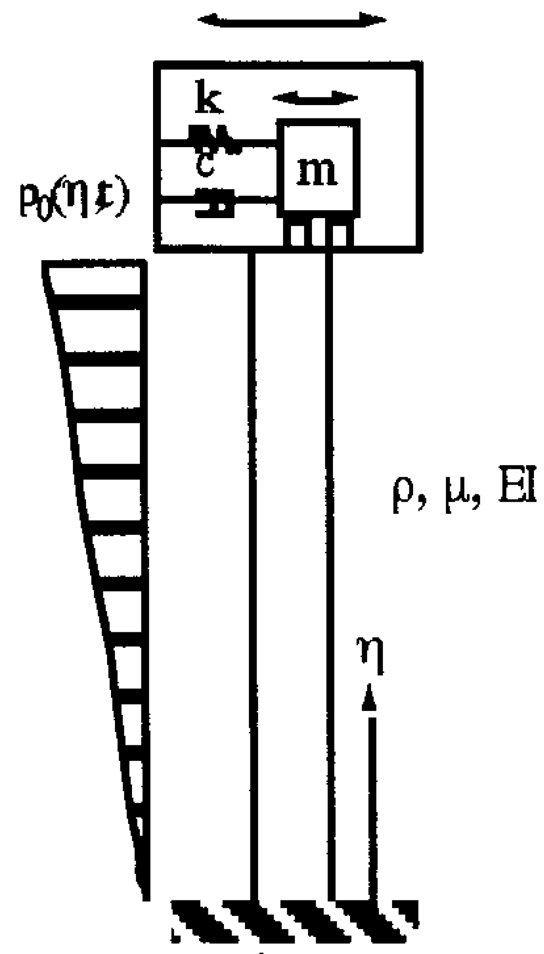


Fig. 2

and defining x_1 is the absolute displacement of the top of the tower and x_2 is the relative displacement of the relative to the deck of the top of the tower. Then the following equation of motion may be derived.

when x_1 is absolute displacement of the tower and

x_2 is relative displacement of the deck

$$T = \underbrace{\frac{1}{2}\left(\frac{21,128}{83,853}\rho L + M\right)\dot{x}_1^2}_{\text{tower \& deck}} + \underbrace{\frac{1}{2} m \left(\dot{x}_1 + x\dot{x}_2\right)^2}_{\text{equipment}}$$

$$V = \underbrace{\frac{1}{2}\left(\frac{2,640}{847}\frac{EI}{L^3}\right)x_1^2}_{\text{tower}} + \underbrace{\frac{1}{2}kx_2^2}_{\text{equipment}}$$

$$Q = \frac{2}{7}Lp_0(t) - \left(\frac{21,128}{83,853}\nu L\right)x\dot{x}_1$$

$\because \delta_2 = 0$, thererfore no viscous force

due to dashpot of equipment

$F = -c\dot{x}_2$

$$\begin{pmatrix} \frac{21,128}{83,853}\rho L + M + m, & m \\ m, & m \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} \frac{21,128}{83,853}\nu L, & 0 \\ 0, & 0c \end{pmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{240}{77}\frac{EI}{L^3}, & 0 \\ 0, & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{2}{7} L p_0(t) \\ 0 \end{Bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{M\ddot{X}} + \boldsymbol{C\dot{X}} + \boldsymbol{KX} = \boldsymbol{F}$$

$\boldsymbol{M}$: mass matrix, $\boldsymbol{C}$: damping matrix, $\boldsymbol{K}$: stiffness matrix, $\boldsymbol{F}$: force matrix

In the above, T.D.O.F was solved, but the other model so-called "M.D.O.F (Many Degrees Of Freedom System)" can be of course solved due to the same method,
n × n matrix as follow may be derived.

$$\begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & m & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \ddot{x} \\ \vdots \\ \vdots \end{pmatrix} + \begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & c & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \dot{x} \\ \vdots \\ \vdots \end{pmatrix} + \begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & k & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ x \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \\ f(t) \\ \vdots \\ \vdots \end{pmatrix}$$

(3) Modal Analysis

It is possible to solve above vibration equation due to "Modal Analysis".

$$\boldsymbol{M\ddot{X}} + \boldsymbol{C\dot{X}} + \boldsymbol{KX} = \boldsymbol{f}(t) \qquad (1)$$

The first, excepting damping matrix and force matrix from above matrix equation.

$$\boldsymbol{M\ddot{X}} + \boldsymbol{KX} = 0 \qquad (2)$$

Assuming solution of Eq. (2)

$$\boldsymbol{X} = \phi e^{i\omega t} \qquad (3)$$

Substituting Eq. (3) into (2)

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) \boldsymbol{\phi} = 0 \qquad (4)$$

$\boldsymbol{\phi} \neq 0$ in order that significant solution exists,
hence coefficient matrix of $\boldsymbol{\phi}$ is 0,
therefore,

$$\det\left(K - \omega^2 M\right) = 0 \qquad (5)$$

ω satisfying Eq. (5) is natural circular frequency ("eigenvalue" in mathematic field), ϕ corresponding to ω is natural frequency mode ("enigen vector"in mathematic field).

ω of which number is (n) is derived by solving Eq. (5) in n-degrees of freedom system,

Each ω is different from the others.

$\omega_1 < \omega_2 < \omega_3 < \ < \omega_n$
ω_1: 1st natural frequency
ω_2: 2nd
⋮
ω_n: nth

substituting solution ω_i into Eq. (2),

$$K\phi - \omega_i^2 M\phi = 0 \quad i = 1,2,3,\cdots,n \qquad (6)$$

hereby

$$\phi = \phi_i \qquad i = 1,2,3,\cdots,n$$

according to orthogonality,

$$\phi_j^T M \phi_i = 0 \quad (i \neq j) \qquad (7)$$

Solution of Eq. (1) is found as follows.
The first, assuming that solution of Eq. (1) is

$$X = \Phi\eta(t) \qquad (8)$$

Φ is modal matrix, which is given by following equation.

$$\Phi = [\phi_1, \phi_2, \cdots, \phi_n] = \begin{bmatrix} \phi_{11}, & \phi_{12}, & \cdots, & \phi_{1n} \\ \phi_{21}, & \phi_{22}, & \cdots, & \phi_{2n} \\ \cdots, & \cdots, & \cdots, & \cdots \\ \phi_{n1}, & \phi_{n2}, & \cdots, & \phi_{nn} \end{bmatrix} : n \times n \text{ matrix} \qquad (9)$$

Each line of matrix Φ is namely constituted as each natural vibration mode.

Substituting Eq. (8) into (1)

$$M\Phi\ddot{\eta} + K\Phi\eta = F \qquad (10)$$

$$\Phi^T M\Phi\ddot{\eta} + \Phi^T K\Phi\eta = \Phi^T F \qquad (11)$$

$$\Phi^T M \Phi = [\phi_1, \phi_2, \cdots, \phi_n]^T M [\phi_1, \phi_2, \cdots, \phi_n] = \begin{bmatrix} \phi_1^T M\phi_1, & \phi_1^T M\phi_2, & \cdots, & \phi_1^T M\phi_n \\ \phi_2^T M\phi_1, & \phi_2^T M\phi_2, & \cdots, & \phi_2^T M\phi_n \\ \cdots, & \cdots, & \cdots, & \cdots \\ \phi_n^T M\phi_1, & \phi_n^T M\phi_2, & \cdots, & \phi_n^T M\phi_n \end{bmatrix}$$

(12)

due to orthogonality, all of nonsymetric elements is 0, hence

$$\boldsymbol{\Phi}^T \boldsymbol{M} \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \quad (13)$$

similarly,

$$\boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1^T \boldsymbol{K} \boldsymbol{\phi}_1, & & & \\ & \boldsymbol{\phi}_2^T \boldsymbol{K} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \boldsymbol{\phi}_n^T \boldsymbol{K} \boldsymbol{\phi}_n, \end{bmatrix} \quad (14)$$

since

$$\boldsymbol{\phi}_i^T \boldsymbol{K} \boldsymbol{\phi}_i = \omega_i^2 \boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_i \,, \quad (15)$$

$$\boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} = \begin{bmatrix} \omega_1^2 \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \omega_2^2 \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \omega^2{}_1 \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \quad (16)$$

therefore

$$\begin{bmatrix} \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \ddot{\boldsymbol{\eta}} + \begin{bmatrix} \omega_1^2 \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \omega_2^2 \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \omega_{n1}^2 \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \boldsymbol{\eta}$$

$$= \boldsymbol{\Phi}^T \boldsymbol{F} \quad (17)$$

Eq. (21) means that quadratic differential Eq. (10) is transformed to non-coupled equation. This is the most characteristic of Modal Analysis.

Solving differential equation

$$\phi_i^T \boldsymbol{M} \phi_i \eta_i + \omega_i^2 \phi_i^T \boldsymbol{M} \phi_i \eta_i = \phi_i^T F \quad i = 1,2,3,\cdots,n \quad (18)$$

substituting solution η_i into Eq. (15), solution $\boldsymbol{X}$ can be found.

In case in which damping matrix is considered, if damping matrix is symmetric as follows,

$$\boldsymbol{\Phi}^T \boldsymbol{C} \boldsymbol{\Phi} = \mathrm{diag}\left(2\beta_1\omega_1, 2\beta_2\omega_2, \cdots, 2\beta_n\omega_n\right) = \begin{pmatrix} 2\beta_1\omega_1 & & & \\ & 2\beta_2\omega_2 & & \\ & & \ddots & \\ & & & 2\beta_n\omega_n \end{pmatrix} \quad (19)$$

Modal Analysis is applicable. In actual analyzing, β_i is assumed so that Eq. (19) may be formed.

Modal Analysis is, as above, theoretically systematic, in which solution can be simultaneously found while considering vibration property, but linear-M.D.O.F. of non-coupled system can be only applied.

When $\boldsymbol{CM}^{-1}\boldsymbol{K} = \boldsymbol{KM}^{-1}\boldsymbol{C}$ (proportional damping), the eigenvalue problem of Eq. (20) may be solved indirectly as follows:

$$(\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{C} + \boldsymbol{K})\boldsymbol{a} = \boldsymbol{0}$$

(substituting $\boldsymbol{X} = \boldsymbol{a}e^{\lambda t}$ into (1))

$$\boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{M} \boldsymbol{M}^{-\frac{1}{2}} \equiv \boldsymbol{I} = \mathrm{diag}(1, \cdots, 1) = \begin{pmatrix} 1 & & & \\ & \ddots & \boldsymbol{0} & \\ & & \ddots & \\ & \boldsymbol{0} & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{K} \boldsymbol{M}^{-\frac{1}{2}} \equiv \overline{\overline{\boldsymbol{A}}}$$

$$(\mu^2 \boldsymbol{I} + \overline{\overline{\boldsymbol{A}}})\boldsymbol{b} = \boldsymbol{0}, \quad (\boldsymbol{b}^T \boldsymbol{b} = \boldsymbol{I})$$

for example, in case of T.D.O.F,

$$\omega_0^2 \begin{pmatrix} \mu^2 + a_{11}, & a_{12} \\ a_{21}, & \mu^2 + a_{22} \end{pmatrix} \boldsymbol{b} = \boldsymbol{0}, \quad \det \begin{pmatrix} \mu^2 + a_{11}, & a_{12} \\ a_{21}, & \mu^2 + a_{22} \end{pmatrix} = \boldsymbol{0}$$

$-\mu^2 =$ m or n, sustituting m,n thereinto

$$\Rightarrow \quad \omega_0^2 \begin{pmatrix} \text{`}\,, & \text{`} \\ \text{`}\,, & \text{`} \end{pmatrix} \boldsymbol{b}_1 = 0, \ \omega_0^2 \begin{pmatrix} \text{`}\,, & \text{`} \\ \text{`}\,, & \text{`} \end{pmatrix} \boldsymbol{b}_2 = 0 \quad \Rightarrow \quad \boldsymbol{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} \ (\rightharpoonup \boldsymbol{B} = (\boldsymbol{b}_1, \boldsymbol{b}_2))$$

$$\omega_1 = \sqrt{-\mu_1^2}\,\omega_0, \ \omega_2 = \sqrt{-\mu_2^2}\,\omega_0$$

$$\Downarrow$$

$$\boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{K} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} = \begin{pmatrix} -\mu_1^2, & 0 \\ 0, & -\mu_2^2 \end{pmatrix}, \quad \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{C} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} = \begin{pmatrix} \tau_1^2, & 0 \\ 0, & \tau_2^2 \end{pmatrix}$$

Substituting $\boldsymbol{X} = \boldsymbol{B}\boldsymbol{\eta}$

$$\boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{M} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} \ddot{\boldsymbol{\eta}} + \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{C} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} \dot{\boldsymbol{\eta}} + \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{K} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} \boldsymbol{\eta} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{f} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B}$$

$$\therefore \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ddot{\boldsymbol{\eta}} + \begin{pmatrix} \tau_1^2 & 0 \\ 0 & \tau_2^2 \end{pmatrix} \dot{\boldsymbol{\eta}} + \begin{pmatrix} -\mu_1^2 & 0 \\ 0 & -\mu_2^2 \end{pmatrix} \boldsymbol{\eta} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{f} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B}$$

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